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The Concave Nontransitive Consumer

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Abstract. The preference of a concave nontransitive consumer is represented by a skew-symmetric and concave-convex bifunction on the set of all commodity bundles. This paper characterizes finite sets of demand observations that are consistent with the demand behavior of such kind of consumer by a generalized monotonicity property.

Key words: Consumer theory, representability, revealed preference, recoverability, generalized monotonicity

1. Introduction

The traditional theory of consumer demand is based on the assumption that a consumer's choice is derived from the utility maximization hypothesis. More precisely, it is assumed that a consumer's taste can be described by a real-valued function u defined on the set $\mathbb{R}_+^\ell$ of all possible consumption bundles of ℓ commodities, i.e. for any $x \in \mathbb{R}_+^\ell$, $u(x)$ is interpreted as the subjective value (utility) that the consumer assigns to the consumption vector x. Given a set of feasible alternatives $X \subseteq \mathbb{R}_+^\ell$, the consumer chooses a bundle $x^* \in X$ such that $u(x^*) \geqslant u(x)$ for all $x \in X$, i.e. x^* maximizes u on X.

How can this hypothesis be tested in a competitive environment? Suppose that the consumer is observed to buy the bundle x at the price vector $p \in \mathbb{R}_{++}^\ell$. Then any commodity vector y that is not more expensive than x is also affordable at p. Thus, if the consumer maximizes a utility function u, y cannot have a higher utility level than x. This argument is extended as follows.

Assume that there are finitely many demand observations (p_i, x_i), i.e. x_i is demanded at prices p_i for $i = 1, \ldots, n$. Then $u : \mathbb{R}_+^\ell \to \mathbb{R}$ *rationalizes* these data, if for $i = 1, \ldots, n$ and every $x \in \mathbb{R}_+^\ell$

$$p_i x_i \geqslant p_i x \text{ implies } u(x_i) \geqslant u(x).$$

Hence, observed behavior is consistent with the utility maximization hypothesis if there exists a rationalizing utility function. The question is whether there are testable conditions for this consistency.

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Of course, there is a trivial answer if no restriction on the utility function is imposed. Obviously, any constant function u rationalizes any given set of observations. Put differently, we look for a nondegenerate rationalization, preferably with some nice additional properties.

A natural requirement for u to be nontrivial is that u is *locally nonsatiated*, i.e., for every $x \in \mathbb{R}^\ell_+$ and every neighborhood N of x, there is some $y \in N$ such that $u(y) > u(x)$. An even stronger condition is obtained by the natural assumption that all commodities are desirable: u is called *monotone* if $x > y$ implies $u(x) > u(y)$ for all $x, y \in \mathbb{R}^\ell_+$.

The following characterization of rationalizability has been stated as Afriat's Theorem in Varian (1982). Its origin is Afriat (1967).

THEOREM 1. *Let $D = \{(p_i, x_i) \mid i \in I\}$ be a finite set of demand observations. Then the following conditions are equivalent:*

(i) There exists a locally nonsatiated utility function that rationalizes the data.

(ii) The observations satisfy the 'Generalized Axiom of Revealed Preference (GARP)', i.e. for any k observations $(p_1, x_1), \ldots, (p_k, x_k) \in D$ the inequalities

$$p_i(x_{i+1} - x_i) \leqslant 0 \text{ for } i = 1, \ldots, k-1$$

imply that $p_k(x_1 - x_k) \geqslant 0$.

(iii) There exist real numbers $u_i, \pi_i > 0$ for every $i \in I$ that satisfy the 'Afriat inequalities'

$$u_i \leqslant u_j + \pi_j p_j(x_i - x_j) \text{ for all } i, j \in I.$$

(iv) There exists a continuous, concave, and monotone utility function that rationalizes the data.

This is a remarkable result. First, it characterizes consistency with the utility maximization hypothesis by the testable conditions (ii) or (iii). Second, it shows that if the data can be rationalized by any locally nonsatiated utility function at all, then they can be rationalized by a utility function with very nice properties.

On the other hand, it is well known that empirical studies have not confirmed that consumers behave as utility maximizers. Put differently, consumers often do not act in accordance with a transitive preference relation.

The aim of this paper is to provide a characterization in the nontransitive case that is analogous to Afriat's theorem. For this purpose we employ the concept of a *nontransitive consumer* as introduced by Shafer (1974). He has shown that preferences which are not necessarily transitive can be numerically represented by a skew-symmetric function $r : \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$. Such a representation generalizes the notion of utility in the sense that r can be defined by $r(x, y) = u(x) - u(y)$ if the consumer's preference is described by a utility function u. Thus, a concave utility function u corresponds to a representation r that is concave in the first argument (resp. convex in the second argument). A consumer whose preference can be represented by such a function will be called a *concave nontransitive consumer*.

In the next section, the notion of a nontransitive consumer is described more precisely. Section 3 contains the main result. It turns out that finitely many observations are consistent with the behavior of a concave nontransitive consumer if and only if they satisfy a generalized monotonicity condition which will be called *monotone transformability* . Finally, Section 4 relates this property to two other concepts of generalized monotonicity that have been introduced by Daniilidis and Hadjisavvas (1999).

2. The nontransitive consumer

Let the preference of a consumer be described by a binary relation R on the set $\mathbb{R}^\ell_+$ of all possible consumption bundles. For $x, y \in \mathbb{R}^\ell_+$, xRy is interpreted as 'x is at least as good as y' or 'x is weakly preferred to y'. It is assumed that the consumer is able to compare any two bundles, i.e., that R is *complete*:

$$\text{For all } x, y \in \mathbb{R}^\ell_+ : xRy \vee yRx.$$

In contrast to the traditional theory, R is not necessarily transitive. Following Shafer (1974), we call the consumer *nontransitive* although it does not mean that transitivity is excluded.

Recall that for a transitive preference R a real-valued function u defined on $\mathbb{R}^\ell_+$ is called a *utility representation* of R, if and only if for all $x, y \in \mathbb{R}^\ell_+$

$$xRy \Leftrightarrow u(x) \geqslant u(y).$$

The following extension to the nontransitive consumer has been introduced by Shafer (1974).

DEFINITION 1. A function $r: \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$ is a numerical *representation* of the preference R if for all $x, y \in \mathbb{R}^\ell_+$:

$$xRy \Leftrightarrow r(x, y) \geqslant 0, \tag{1}$$

$$r(x, y) = -r(y, x). \tag{2}$$

Clearly, if u is a utility representation of R then r_u, defined by $r_u(x, y) = u(x) - u(y)$, represents R in the sense of Definition 1.

It is also obvious that any preference R can be represented. Define the *strict preference* relation P by xPy iff $xRy \wedge \neg yRx$ and the *indifference* relation I by xIy iff $xRy \wedge yRx$. Completeness of R implies that $\mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$ is equal to the disjoint union $P \cup I \cup P^{-1}$. Since $R = P \cup I$, R is represented by the function r which takes the values 1 on P, 0 on I, and -1 on P^{-1}.

On the other hand, any skew-symmetric function $r: \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$ induces a preference R_r that is represented by r (simply define xR_ry by $r(x, y) \geqslant 0$). If r is continuous then R_r is *continuous*, i.e., R_r is closed in $\mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+$. Conversely, as shown by Shafer (1974), any continuous preference has a continuous representation.

These results show that a (continuous) nontransitive consumer can be equivalently described by a complete (continuous) binary relation or by a (continuous) skew-symmetric bifunction on $\mathbb{R}^\ell_+$. Of course, like in traditional utility theory, the relationship is not one-to-one since any sign-preserving transformation of a numerical representation of R also represents R.

The demand theory of the nontransitive consumer in a competitive environment is also analogous to the traditional approach. Assume that the ℓ commodity prices are given by a price vector $p \in \mathbb{R}^\ell_{++}$ and that the consumer's wealth is $w \geqslant 0$. Then the consumer chooses a consumption bundle in the budget set

$$B(p, w) = \{x \in \mathbb{R}^\ell_+ \mid px \leqslant w\}$$

that is weakly preferred to all other bundles in $B(p, w)$. If the preference is represented by r, this is equivalent to the choice of $x^* \in B(p, w)$ such that

$$r(x^*, x) \geqslant 0 \text{ for all } x \in B(p, w).$$

x^* is called an *optimal* commodity bundle or a *demand* vector at the price–wealth pair (p, w). The problem of finding such an x^* has been called an *equilibrium problem* by Blum and Oettli (1994).

It is well known that, in contrast to utility maximization, continuity of r is not sufficient for the existence of an optimal bundle. However, as already shown by Shafer (1974), the problem is solvable if, in addition, r is concave in the first argument or, equivalently, r is convex in the second argument. This and some other important properties are put together in

DEFINITION 2. A representation r is called *concave–convex* if for every $x \in \mathbb{R}^\ell_+$ the function $r(\cdot, x) : \mathbb{R}^\ell_+ \to \mathbb{R}$ is concave, *nonsatiated* if for every $x \in \mathbb{R}^\ell_+$ there exists $y \in \mathbb{R}^\ell_+$ such that $r(y, x) > 0$, *monotone* if for all $x, y, z \in \mathbb{R}^\ell_+ : x > y$ implies $r(x, z) > r(y, z)$.

A consumer is called *concave* (resp. *nonsatiated, monotone*) if his preference representation is concave–convex (resp. nonsatiated, monotone).

It is easy to see that a monotone consumer never chooses a bundle in his budget set $B(p, w)$ that is less expensive than w. Moreover, an optimal bundle x^* in $B(p, w)$ is strictly preferred to all x such that $px < w$. Actually, the same conclusions can be drawn for a nonsatiated, concave consumer as shown by the following

PROPOSITION 1. *Let r be a nonsatiated, concave-convex representation and let x^* be an optimal bundle in $B(p, w)$. Then $px^* = w$ and $r(x^*, x) > 0$ for all x such that $px < w$.*

Proof. Consider $x, z \in \mathbb{R}^\ell_+$ such that $r(x, z) \geqslant 0$. Since r is nonsatiated, there exists $y \in \mathbb{R}^\ell_+$ such that $r(y, z) > 0$. Concavity of r in the first argument implies that $r(\lambda x + (1 - \lambda)y, z) \geqslant \lambda r(x, z) + (1 - \lambda)r(y, z) > 0$ for all $\lambda \in [0, 1[$.

Hence, it follows from $r(x, z) \geqslant 0$ that there are $x' \in \mathbb{R}^\ell_+$ arbitrarily close to x such that $r(x', z) > 0$.

Now the claim is easily proved. If we assume that $px^* < w$, then $r(x^*, x^*) = 0$ implies the existence of x' with $px' < w$ and $r(x', x^*) > 0$, a contradiction to the optimality of x^*.

In order to prove the second part, assume that $px < w$ and $r(x^*, x) \leqslant 0$ or, equivalently, $r(x, x^*) \geqslant 0$. Again, the existence of x' with $px' < w$ and $r(x', x^*) > 0$ contradicts the optimality of x^*. □

3. The main result

Assume that a consumer demands the commodity bundle x_i at the price vector $p_i, i = 1, \ldots, n$. When are these n demand observations consistent with a preference maximizing behavior of a concave consumer?

In order to be more precise, we give the following

DEFINITION 3. A preference representation r *rationalizes* the set of observations $\{(p_i, x_i) \mid i = 1, \ldots, n\}$ if for all i and all $x \in \mathbb{R}^\ell_+$

$$p_i x_i \geqslant p_i x \text{ implies } r(x_i, x) \geqslant 0.$$

The existence of a rationalizing concave-convex representation r is of course trivial if no further properties of r are required. Indeed, if the consumer is indifferent between all bundles, i.e., $r(x, y) = 0$ for all x, y, then any set of observations is rationalized by r. To exclude such a degenerate case, a natural requirement is that the preference should be nonsatiated. This is taken into account by the next result.

THEOREM 2. *Let $\{(p_i, x_i) \mid i = 1, \ldots, n\}$ be a finite set of demand observations. Then the following conditions are equivalent:*

(i) There exists a nonsatiated, concave-convex representation that rationalizes the observations.

(ii) For any set of real numbers $\lambda_{ij} \geqslant 0 (1 \leqslant i, j \leqslant n)$ such that $\lambda_{ij} = \lambda_{ji}$, the inequalities

$$\sum_{j=1}^{n} \lambda_{ij} p_i (x_j - x_i) \leqslant 0 \textit{ for } i = 1, \ldots, n$$

imply the equalities

$$\sum_{j=1}^{n} \lambda_{ij} p_i (x_j - x_i) = 0 \textit{ for } i = 1, \ldots, n.$$

(iii) There exist real numbers $\pi_i > 0 (1 \leqslant i \leqslant n)$ such that for all $i, j \in \{1, \ldots, n\}$:

$$(\pi_i p_i - \pi_j p_j)(x_i - x_j) \leqslant 0.$$

(iv) There exist real numbers $\rho_{ij}(1 \leqslant i, j \leqslant n)$ *with* $\rho_{ji} = -\rho_{ij}$ *and* $\pi_i > 0(1 \leqslant i \leqslant n)$ *such that for all* $i, j \in \{1, \ldots, n\}$:

$$\rho_{ij} \leqslant \pi_j p_j(x_i - x_j).$$

(v) There exists a continuous, monotone, concave-convex representation that rationalizes the observations.

Proof. (i) $\Rightarrow$ (ii): It will be shown that a violation of (ii) leads to a contradiction. If (ii) does not hold, then there are $\lambda_{ij} \geqslant 0$ with $\lambda_{ij} = \lambda_{ji}$ such that $\sum_{j=1}^n \lambda_{kj} p_k(x_j - x_k) < 0$ for some k and $\sum_{j=1}^n \lambda_{ij} p_i(x_j - x_i) \leqslant 0$ for all $i \neq k$. Define $I = \{i \mid \sum_{j=1}^n \lambda_{ij} \neq 0\}$ and $\lambda'_{ij} = \lambda_{ij} / \sum_{j=1}^n \lambda_{ij}$ for $i \in I$. Since $\sum_{j=1}^n \lambda'_{ij} = 1$, the convex combinations $\bar{x}_i = \sum_{j=1}^n \lambda'_{ij} x_j$ are elements in $\mathbb{R}_+^\ell$. It follows that $p_i(\bar{x}_i - x_i) \leqslant 0$ for all $i \in I$ and $p_k(\bar{x}_k - x_k) < 0$.

If r rationalizes the observations then $r(\bar{x}_i, x_i) \leqslant 0$ for all $i \in I$ and, since $p_k\bar{x}_k < p_k x_k$, $r(\bar{x}_k, x_k) < 0$ by Proposition 1. Concavity of r in the first argument implies

$$\sum_{j=1}^n \lambda'_{ij} r(x_j, x_i) \leqslant 0 \text{ and } \sum_{j=1}^n \lambda'_{kj} r(x_j, x_k) < 0.$$

Hence, $\sum_{j=1}^n \lambda_{ij} r(x_j, x_i) \leqslant 0$ for $i \in I$ and $\sum_{j=1}^n \lambda_{kj} r(x_j, x_k) < 0$. Taking into account that $\sum_{j=1}^n \lambda_{ij} r(x_j, x_i) = 0$ for $i \notin I$, we obtain

$$\sum_{i=1}^n \left(\sum_{j=1}^n \lambda_{ij} r(x_j, x_i) \right) = \sum_{i,j=1}^n \lambda_{ij} r(x_j, x_i) < 0.$$

On the other hand, $\lambda_{ij} = \lambda_{ji}$ and $r(x_j, x_i) = -r(x_i, x_j)$ implies $\sum_{i,j=1}^n \lambda_{ij} r(x_j, x_i) = 0$, i.e., a contradiction has been derived.

(ii)$\Rightarrow$(iii): For $i, j \in \{1, \ldots, n\}$, $i < j$, define $a^{ij} \in \mathbb{R}^n$ by

$$a_k^{ij} = \begin{cases} p_i(x_j - x_i) & k = i \\ p_j(x_i - x_j) & k = j \\ 0 & k \neq i, j. \end{cases}$$

Consider the closed convex cone in $\mathbb{R}^n$ that is generated by these vectors a^{ij}, i.e., the set $C = \{\sum_{i<j} \mu_{ij} a^{ij} \mid \mu_{ij} \geqslant 0, i < j\}$. Observe that the k-th component of an element of C is given by

$$\begin{aligned} \sum_{i<j} \mu_{ij} a_k^{ij} &= \sum_{k<j} \mu_{kj} p_k(x_j - x_k) + \sum_{i<k} \mu_{ik} p_k(x_i - x_k) \\ &= \sum_{j=1}^n \lambda_{kj} p_k(x_j - x_k), \end{aligned}$$

where $\lambda_{kj} = \mu_{kj}$ for $k < j$, $\lambda_{kj} = \mu_{jk}$ for $j < k$, and $\lambda_{kj} = 0$ for $k = j$. Hence, $\lambda_{kj} = \lambda_{jk}$ for $k, j \in \{1, \dots, n\}$.

Condition (ii) guarantees that $C \cap \mathbb{R}^n_- = \{0\}$. By a well-known separation theorem for closed convex cones (see, e.g., Nikaido (1968), Theorem 3.6), there exists $\pi \in \mathbb{R}^n_{++}$ such that $\pi a^{ij} \geqslant 0$ for all a^{ij}. Since

$$\pi a^{ij} = \pi_i p_i(x_j - x_i) + \pi_j p_j(x_i - x_j) = -(\pi_i p_i - \pi_j p_j)(x_i - x_j),$$

we have proved (iii).
(iii)⇒(iv): Assume that (iii) holds and define

$$\rho_{ij} = \frac{1}{2}\left[\pi_i p_i(x_i - x_j) - \pi_j p_j(x_j - x_i)\right].$$

Obviously, $\rho_{ij} = -\rho_{ji}$ for all i, j. Furthermore, we obtain

$$\begin{aligned}\rho_{ij} =& \frac{1}{2}\left[\pi_i p_i(x_i - x_j) + \pi_j p_j(x_i - x_j)\right]\\ =& \frac{1}{2}\left[\pi_i p_i(x_i - x_j) - \pi_j p_j(x_i - x_j) + 2\pi_j p_j(x_i - x_j)\right].\end{aligned}$$

By (iii), $\pi_i p_i(x_i - x_j) - \pi_j p_j(x_i - x_j) = (\pi_i p_i - \pi_j p_j)(x_i - x_j) \leqslant 0$. Hence, $\rho_{ij} \leqslant \pi_j p_j(x_i - x_j)$ for all i, j.
(iv)⇒(v): Let (iv) be satisfied and define for $x, y \in \mathbb{R}^\ell_+$

$$r_{ij}(x, y) = \rho_{ij} + \pi_i p_i(x - x_i) - \pi_j p_j(y - x_j).$$

Since $\rho_{ji} = -\rho_{ij}$, it follows that

$$-r_{ij}(x, y) = \rho_{ji} + \pi_j p_j(y - x_j) - \pi_i p_i(x - x_i) = r_{ji}(y, x).$$

Now let $\Delta = \{\lambda \in \mathbb{R}^n_+ \mid \sum_{i=1}^n \lambda_i = 1\}$ be the $(n-1)$-dimensional standard simplex and let $r : \mathbb{R}^\ell_+ \times \mathbb{R}^\ell_+ \to \mathbb{R}$ be defined by

$$r(x, y) := \min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, y) = \max_{\mu\in\Delta} \min_{\lambda\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, y),$$

where the latter equality is valid by the famous Minimax Theorem of von Neumann (1928). It will be shown that r fulfills the requirements stated in condition (v).

Since $-r_{ij}(x, y) = r_{ji}(y, x)$, we obtain

$$\begin{aligned}-r(x, y) &= -\min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, y) = \max_{\lambda\in\Delta} \min_{\mu\in\Delta} \sum_{i,j} -\lambda_i \mu_j r_{ij}(x, y)\\ &= \max_{\lambda\in\Delta} \min_{\mu\in\Delta} \sum_{j,i} \mu_j \lambda_i r_{ji}(y, x) = r(y, x),\end{aligned}$$

i.e., r is skew-symmetric.

Continuity of r follows by applying twice the maximum theorem in Berge (1997), Chapter VI, §3. Since $r_{ij}(x, y)$ is continuous in x and y for all i and j, the function $F : \Delta^2 \times \mathbb{R}^l_+ \times \mathbb{R}^l_+ \to \mathbb{R}$ defined by

$$F(\lambda, \mu, x, y) = \sum_{i,j} \lambda_i \mu_i r_{ij}(x, y)$$

is also continuous. By the maximum theorem, $\max_{\mu\in\Delta} F(\lambda, \mu, x, y) = G(\lambda, x, y)$ is continuous in $\lambda, x,$ and y. Applying the maximum theorem once again to $-G$ yields the continuity of $\min_{\lambda\in\Delta} G(\lambda, x, y) = r(x, y)$ with respect to x and y.

In order to prove that r is monotone, consider $x, y, z \in \mathbb{R}^\ell_+$ such that $x > y$. Since all price vectors p_i are strictly positive, $r_{ij}(x, z) > r_{ij}(y, z)$. This implies $\max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, z) > \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(y, z)$ for all $\lambda \in \Delta$ and thus

$$\min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, z) > \min_{\lambda\in\Delta} \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(y, z).$$

Hence, $r(x, z) > r(y, z)$, i.e. r is monotone.

Consider now for fixed $y \in \mathbb{R}^\ell_+$ and $\lambda \in \Delta$ the function $r_\lambda : \mathbb{R}^\ell_+ \to \mathbb{R}$ defined by

$$r_\lambda(x) = \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x, y).$$

$$\begin{aligned}
\text{Then } r_\lambda(x) &= \max_{\mu\in\Delta} \sum_i \lambda_i \sum_j [\mu_j \rho_{ij} + \mu_j \pi_i p_i(x - x_i) - \mu_j \pi_j p_j(y - x_j)] \\
&= \max_{\mu\in\Delta} \sum_i \lambda_i [\pi_i p_i(x - x_i) + \sum_j \mu_j(\rho_{ij} - \pi_j p_j(y - x_j))] \\
&= \max_{\mu\in\Delta} [\sum_i \lambda_i \pi_i p_i(x - x_i) + \sum_{i,j} \lambda_i \mu_j(\rho_{ij} - \pi_j p_j(y - x_j))] \\
&= \sum_i \lambda_i \pi_i p_i(x - x_i) + \max_{\mu\in\Delta} \sum_{i,j} \lambda_i \mu_j(\rho_{ij} - \pi_j p_j(y - x_j)),
\end{aligned}$$

i.e., r_λ is an affine function of x. Hence, $r(x, y) = \min_{\lambda\in\Delta} r_\lambda(x)$ is concave in x since it is a minimum of affine functions of x.

It remains to show that r rationalizes the demand observations $\{(p_i, x_i) \mid i = 1, \ldots, n\}$. Consider an arbitrary observation (p_k, x_k) and $y \in \mathbb{R}^\ell_+$ such that $p_k x_k \geqslant p_k y$. It has to be proved that $r(x_k, y) \geqslant 0$. By definition,

$$\begin{aligned}
r(x_k, y) &= \max_{\mu\in\Delta} \min_{\lambda\in\Delta} \sum_{i,j} \lambda_i \mu_j r_{ij}(x_k, y) \\
&\geqslant \min_{\lambda\in\Delta} \sum_{i,j} \lambda_i \mu^k_j r_{ij}(x_k, y) = \min_{\lambda\in\Delta} \sum_j \lambda_i r_{ik}(x_k, y),
\end{aligned}$$

where $\mu_j^k = 0$ for $j \neq k$ and $\mu_k^k = 1$.

Thus, $r(x_k, y) \geqslant 0$ if $r_{ik}(x_k, y) \geqslant 0$ for all i. By definition, $r_{ik}(x_k, y) = \rho_{ik} + \pi_i p_i(x_k - x_i) - \pi_k p_k(y - x_k)$. By assumption, $\pi_k p_k(y - x_k) \leqslant 0$, and, by (iv), $\rho_{ki} \leqslant \pi_i p_i(x_k - x_i)$ or, equivalently, $\rho_{ik} + \pi_i p_i(x_k - x_i) = -\rho_{ki} + \pi_i p_i(x_k - x_i) \geqslant 0$. Hence, $r_{ik}(x_k, y) \geqslant 0$ for all i.

(v)⇒(i): trivial. □

As in Theorem 1, violations of continuity and monotonicity cannot be detected by a finite number of observations. Indeed, the equivalence between (i) and (v) shows that rationalizability by any nontrivial concave–convex representation implies rationalizability by one that is also continuous and monotone. Conditions (ii), (iii), and (iv) provide testable conditions on the consistency of the data with a preference maximizing behavior of a concave nontransitive consumer. While (iv) is the analogue to (iii) of Theorem 1, (ii) or (iii) replace GARP. However, in contrast to the latter property, it is difficult to give a revealed preference interpretation of these conditions. In view of (iii), the following definition seems to be appropriate.

DEFINITION 4. A finite set $\{(p_i, x_i) \mid i = 1, \ldots, n\}$ of demand observations is called *monotone transformable* if one of the equivalent conditions (ii),(iii),(iv) in Theorem 2 is satisfied.

In the next section, this property will be related to other notions of generalized monotonicity.

4. Some relationships

Consider an arbitrary, not necessarily finite, *demand relation* $D \subseteq \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_+$, which obviously generalizes the case of finitely many demand observations. Recall the following

DEFINITION 5. A demand relation D is called

(1) *pseudomonotone*, if for all $(p, x), (q, y) \in D$

$$p(y - x) \leqslant 0 \text{ implies } q(y - x) \leqslant 0,$$

(2) *properly pseudomonotone* , if for any $(p_1, x_1), \ldots, (p_n, x_n) \in D$ and any $x = \sum_{j=1}^n \lambda_j x_j$ with $\sum_{j=1}^n \lambda_j = 1$ and $\lambda_j > 0$ $(1 \leqslant j \leqslant n)$ the inequalities

$$p_i(x - x_i) \leqslant 0 \text{ for } i = 1, \ldots, n$$

imply the equalities

$$p_i(x - x_i) = 0 \text{ for } i = 1, \ldots, n,$$

(3) *cyclically pseudomonotone*, if for any $(p_1, x_1), \ldots, (p_n, x_n) \in D$ the inequalities

$$p_i(x_{i+1} - x_i) \leqslant 0 \text{ for } i = 1, \ldots, n - 1$$

imply that $p_n(x_1 - x_n) \geqslant 0$.

Notice that, in contrast to most of the literature, these definitions generalize the notion of a *nonincreasing* (instead of nondecreasing) real valued function of one variable. While (1) is a well known standard concept, (2) and (3) have been introduced by Daniilidis and Hadjisavvas (1999). They have shown that (1) and (2) are equivalent if the set $X = \{x \in \mathbb{R}^\ell_+ \mid \exists p \in \mathbb{R}^\ell_{++} : (p, x) \in D\}$ is convex. Notice also that in the case of a finite $D = \{(p_1, x_1), ..., (p_n, x_n)\}$ the implication in (2) has to be satisfied for *all* finite subsets of D, i.e., it is not sufficient to consider only proper convex combinations of $x_1, ..., x_n$.

It can be easily seen that a monotone transformable finite demand relation is pseudomonotone. Indeed, if $(p_i, x_i), (p_j, x_j)$ are two observations such that $p_j(x_i - x_j) \leqslant 0$, then, by condition (iii) in Theorem 2, it follows that $\pi_i p_i(x_i - x_j) \leqslant \pi_j p_j(x_i - x_j) \leqslant 0$. Hence, since $\pi_i > 0$, $p_i(x_i - x_j) \leqslant 0$.

A stronger result is given by

PROPOSITION 2. *Let $D = \{(p_i, x_i) \mid i = 1, \ldots, n\}$ be a finite demand relation.*
(1) If D is cyclically pseudomonotone, then D is monotone transformable.
(2) If D is monotone transformable, then D is properly pseudomonotone.

Proof. (1) Obviously, cyclic pseudomonotonicity of D is equivalent to condition (ii) in Theorem 1. By equivalence with (iii) of Theorem 1, there are $u_i, \pi_i > 0$ $(i = 1, \ldots, n)$ such that $u_i \leqslant u_j + \pi_i p_j(x_i - x_j)$ for all i, j. Defining $\rho_{ij} = u_i - u_j$ yields condition (iv) of Theorem 2, i.e. D is monotone transformable.

(2) It will be shown that condition (ii) of Theorem 2 implies proper pseudomonotonicity of D.

In order to prove that D is properly pseudomonotone, assume that for $J \subseteq \{1, \ldots, n\}$, $x = \sum_{j \in J} \lambda_j x_j$ with $\sum_{j \in J} \lambda_j = 1$ and $\lambda_j > 0$ $(j \in J)$

$$p_i(x - x_i) = \sum_{j \in J} \lambda_j p_i(x_j - x_i) \leqslant 0 \text{ for } i \in J.$$

Setting $\lambda_i = 0$ for $i \in \{1, \ldots, n\}$, $i \notin J$, this implies

$$\lambda_i \sum_{j \in J} \lambda_j p_i(x_j - x_i) = \sum_{j=1}^{n} \lambda_i \lambda_j p_i(x_j - x_i) \leqslant 0$$

for $i = 1, \ldots, n$.

If we define $\lambda_{ij} = \lambda_i \lambda_j$ for $i, j \in \{1, \ldots, n\}$, then $\lambda_{ij} = \lambda_{ji}$ and, by (ii) of Theorem 2, it follows that

$$\sum_{j=1}^{n} \lambda_i \lambda_j p_i(x_j - x_i) = \lambda_i \sum_{j \in J} \lambda_j p_i(x_j - x_i) = 0$$

for $i = 1, \ldots, n$.

Since $\lambda_i > 0$ for $i \in J$, we finally obtain

$$\sum_{j \in J} \lambda_j p_i(x_j - x_i) = p_i(x - x_i) = 0$$

for $i \in J$, i.e. D is properly pseudomonotone. □

We conclude by giving two examples showing that the reverse implications of (1) and (2) in Proposition 2 do not hold, i.e. that monotone transformability is actually different from cyclic pseudomonotonicity as well as from proper pseudomonotonicity.

EXAMPLE 1. Consider the following three demand observations for three commodities:

$$(p_1, x_1) = ((2, 1, 3), (1, 2, 2)),$$
$$(p_2, x_2) = ((3, 2, 1), (2, 1, 2)),$$
$$(p_3, x_3) = ((1, 3, 2), (2, 2, 1)).$$

These observations are monotone transformable (even monotone), since

$$(p_1 - p_2)(x_1 - x_2) = (-1, -1, 2)(-1, 1, 0) = 0$$
$$(p_2 - p_3)(x_2 - x_3) = (2, -1, -1)(0, -1, 1) = 0$$
$$(p_3 - p_1)(x_3 - x_1) = (-1, 2, -1)(1, 0, -1) = 0$$

On the other hand, there is a cycle given by

$$p_2(x_1 - x_2) = p_3(x_2 - x_3) = p_1(x_3 - x_1) = -1,$$

i.e., the observations are not cyclically pseudomonotone. □

The second example is less obvious. Indeed, it can be shown that proper pseudomonotonicity implies monotone transformability if there are not more than three observations.

EXAMPLE 2. Consider the following four demand observations for four commodities:

$$(p_1, x_1) = ((2, 1, 1, 4), (1, 2, 2, 2)),$$
$$(p_2, x_2) = ((4, 2, 1, 1), (2, 1, 2, 2)),$$
$$(p_3, x_3) = ((1, 4, 2, 1), (2, 2, 1, 2)),$$
$$(p_4, x_4) = ((1, 1, 4, 2), (2, 2, 2, 1)).$$

The matrix $A = (a_{ij})_{i,j=1,\dots,4}$ with $a_{ij} = p_i(x_j - x_i)$ is easily calculated and given by

$$A = \begin{pmatrix} 0 & 1 & 1 & -2 \\ -2 & 0 & 1 & 1 \\ 1 & -2 & 0 & 1 \\ 1 & 1 & -2 & 0 \end{pmatrix}.$$

By setting $\lambda_{ij} = \lambda_{ji} = 1$ for $\{i, j\} = \{1, 2\}, \{2, 3\}, \{3, 4\}, \{1, 4\}$ and $\lambda_{ij} = 0$ otherwise, it follows that

$$\sum_{j=1}^{4} \lambda_{ij} a_{ij} = -1 \quad \text{for } i = 1, \ldots, 4.$$

Thus, condition (ii) of Theorem 2 is violated, i.e. the observations are not monotone transformable.

In order to prove that they are properly pseudomonotone, it has to be shown that for arbitrary $\lambda_1, \ldots, \lambda_4 \geqslant 0$ such that $\sum_{i=1}^{4} \lambda_i = 1$ the inequalities

$$\lambda_i \sum_{j=1}^{4} \lambda_j a_{ij} \leqslant 0 \quad \text{for } i =, \ldots, 4$$

imply the equalities

$$\lambda_i \sum_{j=1}^{4} \lambda_j a_{ij} = 0 \quad \text{for } i = 1, \ldots, 4.$$

Assume the inequalities to be satisfied but that $\lambda_i \sum_{j=1}^{4} \lambda_j a_{ij} < 0$ for some i. Without loss of generality, let $i = 1$, i.e., $\lambda_1 \sum_{j=1}^{4} \lambda_j a_{1j} = \lambda_1(\lambda_2 + \lambda_3 - 2\lambda_4) < 0$. This implies $\lambda_1 > 0$ and $\lambda_2 + \lambda_3 < 2\lambda_4$, hence, $\lambda_4 > 0$. Since $\lambda_4 \sum_{j=1}^{4} \lambda_j a_{4j} = \lambda_4(\lambda_1 + \lambda_2 - 2\lambda_3) \leqslant 0$, it follows that $0 < \lambda_1 + \lambda_2 \leqslant 2\lambda_3$, i.e. $\lambda_3 > 0$. By applying the same argument to the third inequality, we obtain $\lambda_2 > 0$. Since all $\lambda_i > 0$, $\sum_{j=1}^{4} \lambda_j a_{ij} \leqslant 0$ for all i, with a strict inequality for at least one i.

However, adding up these inequalities yields $0 = (\lambda_2 + \lambda_3 - 2\lambda_4) + (-2\lambda_1 + \lambda_3 + \lambda_4) + (\lambda_1 - 2\lambda_2 + \lambda_4) + (\lambda_1 + \lambda_2 - 2\lambda_3) < 0$.

Thus, the assumption that $\lambda_i \sum_{j=1}^{4} \lambda_j a_{ij} < 0$ for some i has led to a contradiction, i.e., all inqualities are satisfied as equalities. □

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